Three Dimensional Turing patterns and Equilibrium Concentration Surfaces

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(joint work with B. Kazmierczak, M. Alber, H.G.E. Hentschel and S.A. Newman)

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Outline

1 Stability of Turing Patterns

Turing patterns: Set up

2D: stripes vs. spots

Higher dimensions

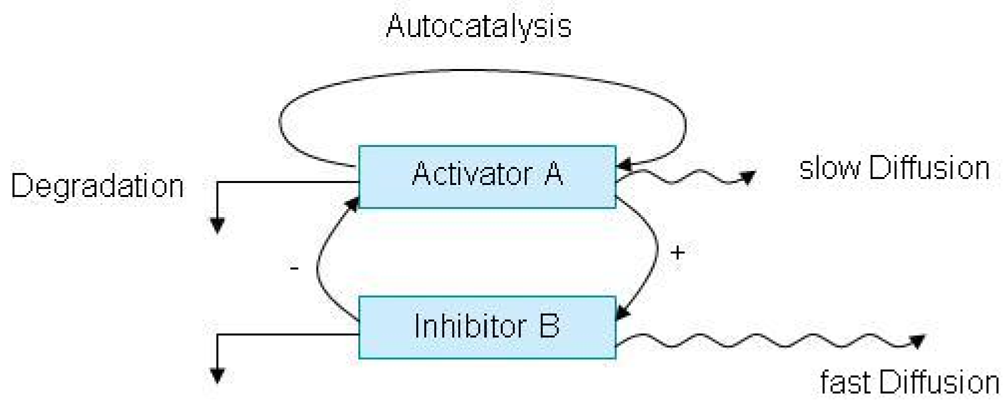
Outline

STABILITY OF TURING PATTERNS

Turing patterns

Turing patterns

Reaction-Diffusion System



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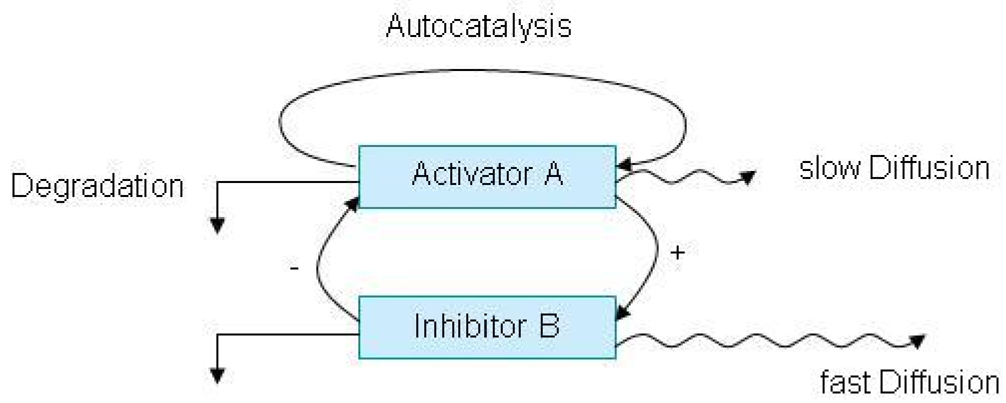

Turing (1952): Such systems can spontaneously give rise to patterns.

Turing patterns in modeling of self-organization phenomena in biology

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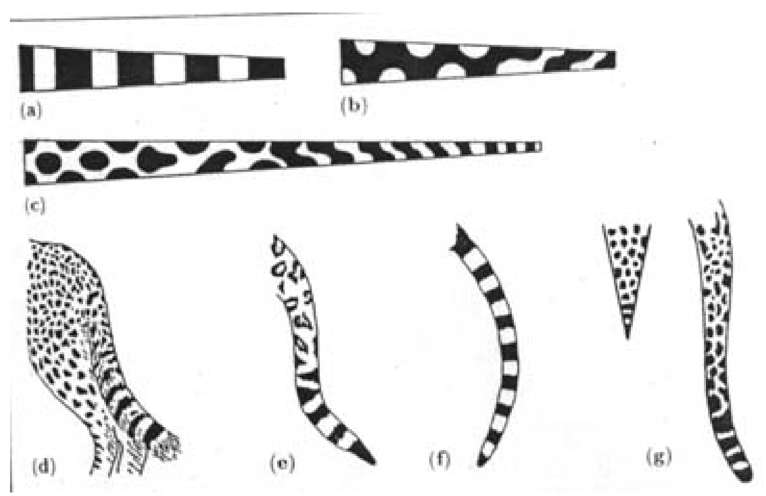

(from J. Murray *Mathematical Biology*)

Turing Bifurcation: Set up

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System of m diffusing and reacting chemicals

$$\frac{\partial \mathbf{U}}{\partial t} = D\nabla^2 \mathbf{U} + \mathcal{F}(\mathbf{U}, \widetilde{\lambda}).$$

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Suppose $\mathbf{U_0} = const$ is a steady state for all $\tilde{\lambda}$: $\mathcal{F}(\mathbf{U_0}, \tilde{\lambda}) = 0$. Consider

$$\frac{\partial u}{\partial t} = (A + D\nabla^2)u + \mathcal{Q}(u,u) + \mathcal{C}(u,u,u) + \tilde{\lambda}\mathcal{B}u + \text{h.o.t} \qquad (1)$$

Here $A = \frac{\partial \mathcal{F}}{\partial \mathbf{U}}(\mathbf{U}_0, 0), \quad u = \mathbf{U} - \mathbf{U}_0.$

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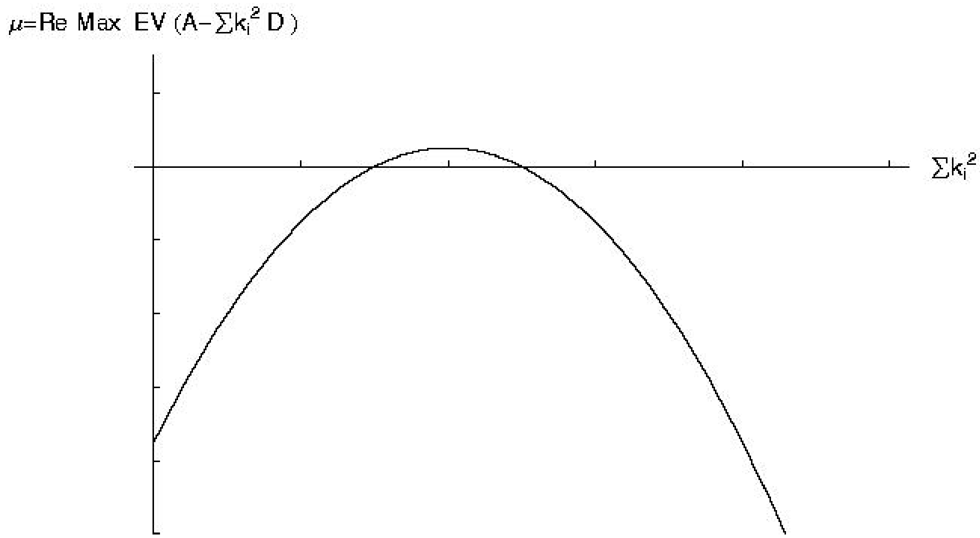

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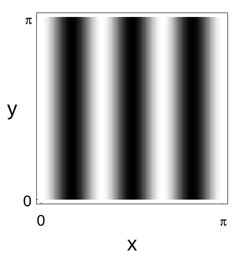
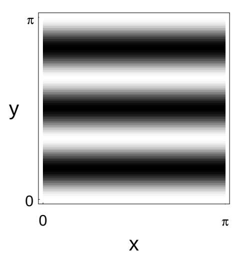
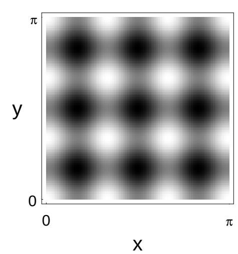
Eg., $k^2 = 6^2$:



$\cos 6x$



$\cos 6y$



$\cos 6x + \cos 6y$

Set up

Suppose we have a Turing bifurcation on the cube $[0, \pi]^n$ at $\widetilde{\lambda} = 0$ for $k^2 = 1$, i.e.:

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Write for steady state bifurcating solution branch
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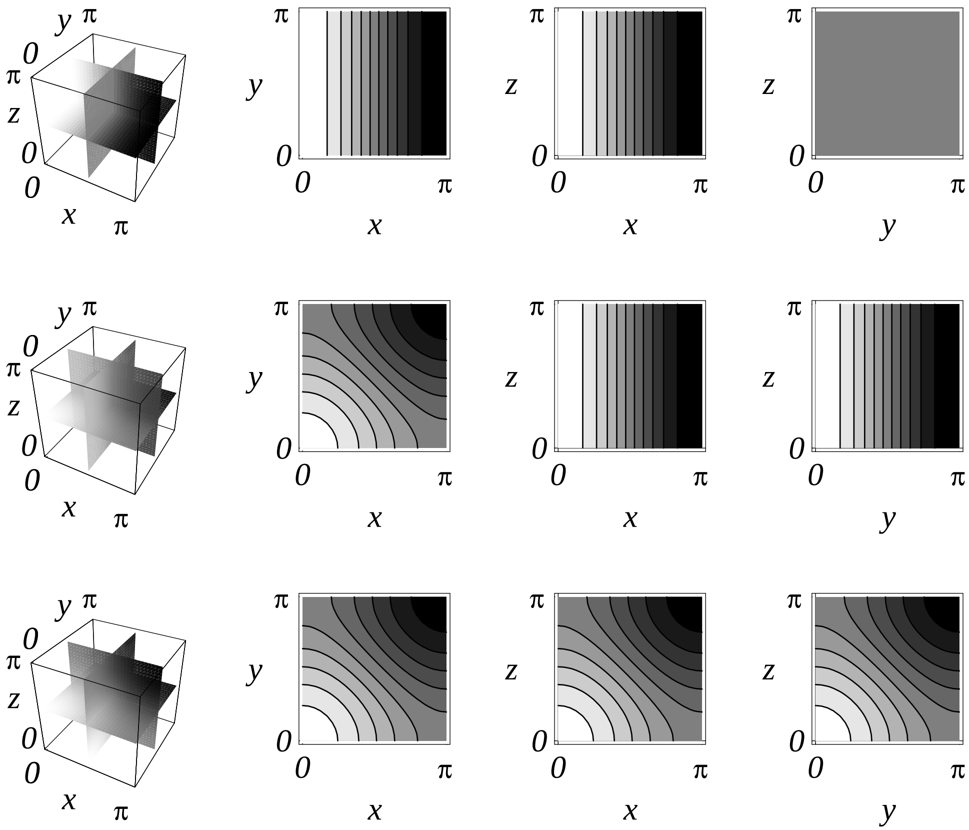
Theorem(Ermentrout 1991) In 2D, spots or stripes can be stable for certain parameter ranges, but not at the same time.

Situation in 3D



$\cos x$ ($\frac{1}{2}$-*sheet*)

$\cos x + \cos y$ ($\frac{1}{2}$-*bar*)

$\cos x + \cos y + \cos z$ ($\frac{1}{2}$-*nodule*)

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1. There is an integer p $(1 \le p \le n)$ such that

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2. The stability of u_ε is determined as follows:
   (a) $\underline{p = 1}$: u_ε is stable iff $b < a < 0$
   (b) $\underline{p = n}$: u_ε is stable iff $a < \min\{b, -(n-1)b\}$
   (c) $\underline{1 < p < n}$: u_ε is *always* unstable.

EQUILIBRIUM CONCENTRATION SURFACES

IN 3-DIMENSIONAL TURING PATTERNS

Brief Reminder: Minimal Surfaces in $\mathbb{R}^3$

Minimal Surfaces are critical points of the Area functional $S \mapsto \mathsf{Area}\mathsf{S} = \int_{\mathsf{S}} \mathsf{d}\mathsf{S}$.

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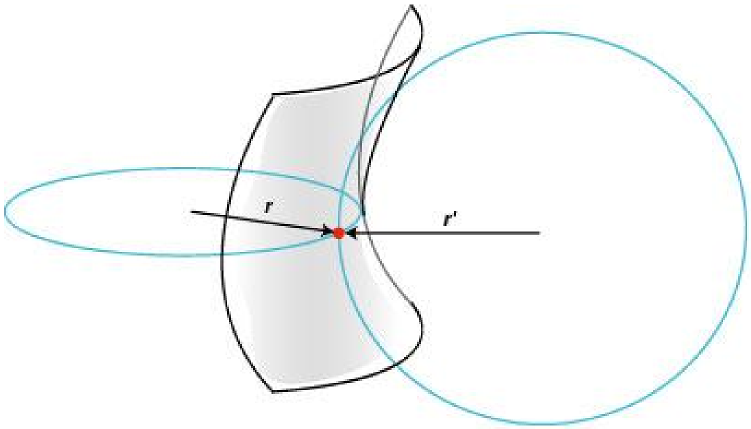

$$\kappa_1 = 1/r, \kappa_2 = 1/r'$$

Definition

Recall: Turing steady state pattern $\mathbf{U} = (U_1, \ldots, U_m)$ satisfies

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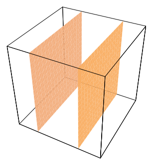
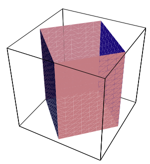
They are the interfaces between regions of high and low concentrations.

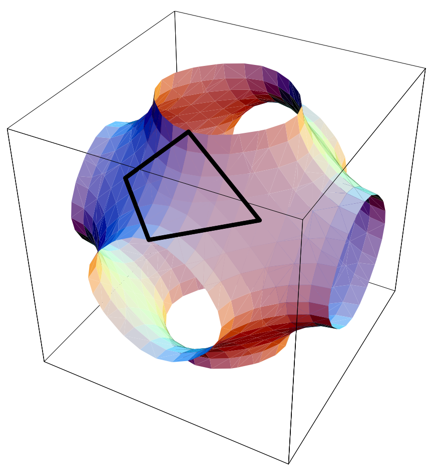
Examples: Turing patterns close to the equilibrium



lamellae

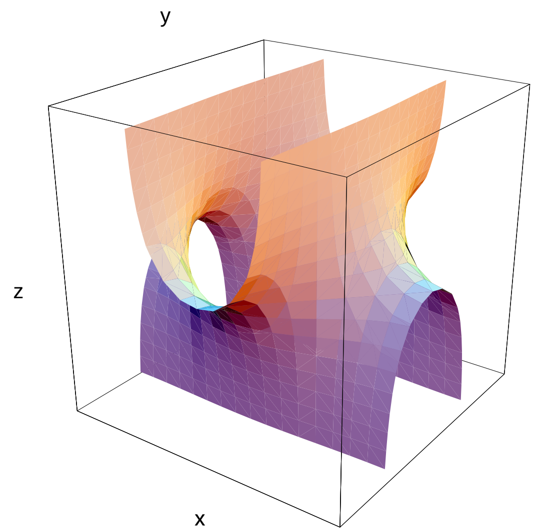


cylinders



nodule – close to Schwarz’ P-surface

Examples: Turing pattern far from equilibrium



Scherk's surface

reported numerically by De Wit, Borckmans, Dewel (1997), Leppaänen et al. (2004)

Variational Principles for Equilibrium Concentration Surfaces I

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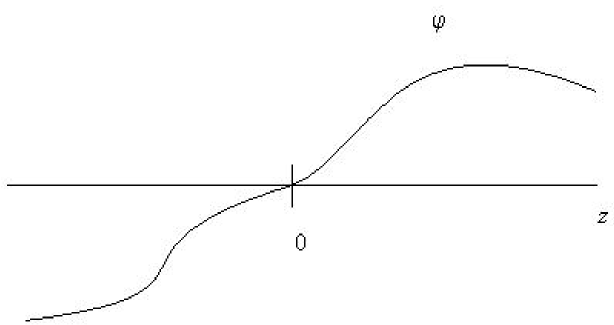

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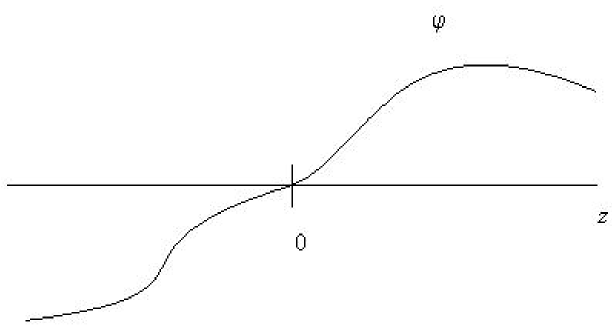

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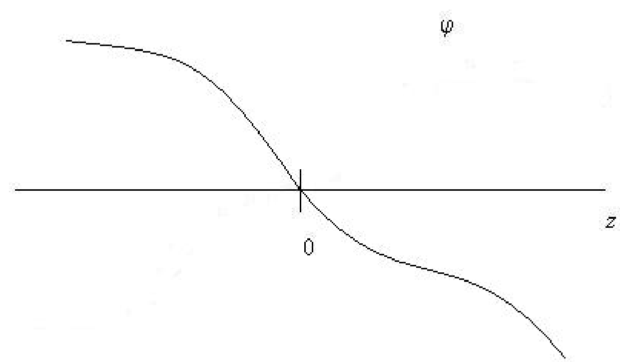

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2) If $U(x) = U^0 + \varepsilon\overline{u}(x) \cdot \mathbf{b} + \cdots$ is an expansion of a Turing pattern *close to the Turing bifurcation*, then $\overline{u}$ is an eigenfunction of the Laplacian, i.e. $\nabla^2\overline{u} = -k^2\overline{u}$ for some k^2.

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UPSHOT: The following variational principles apply **exactly** for certain classes of reaction kinetics, and **to first order** for all Turing patterns close to the Turing bifurcation.

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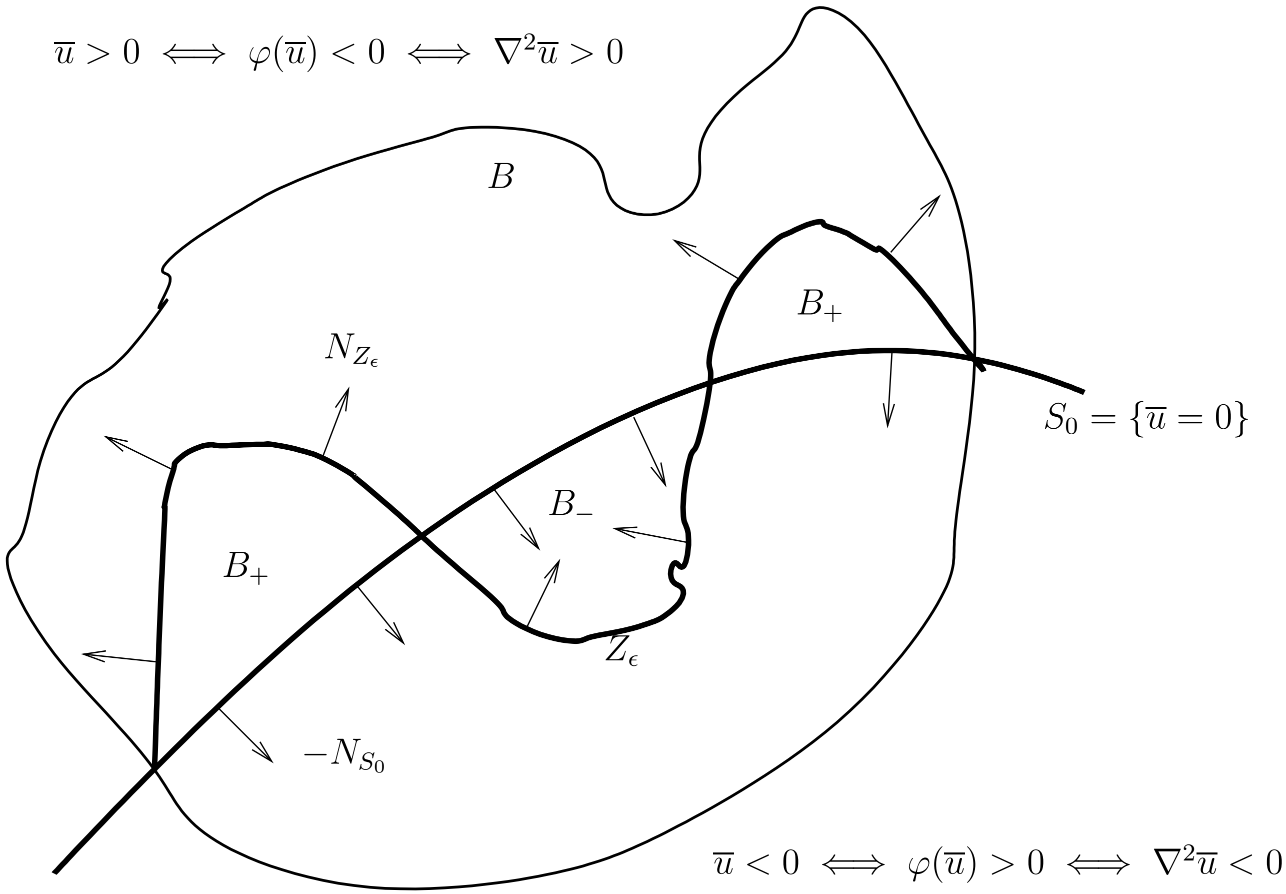
Sketch (for $\varphi'(0) < 0$)



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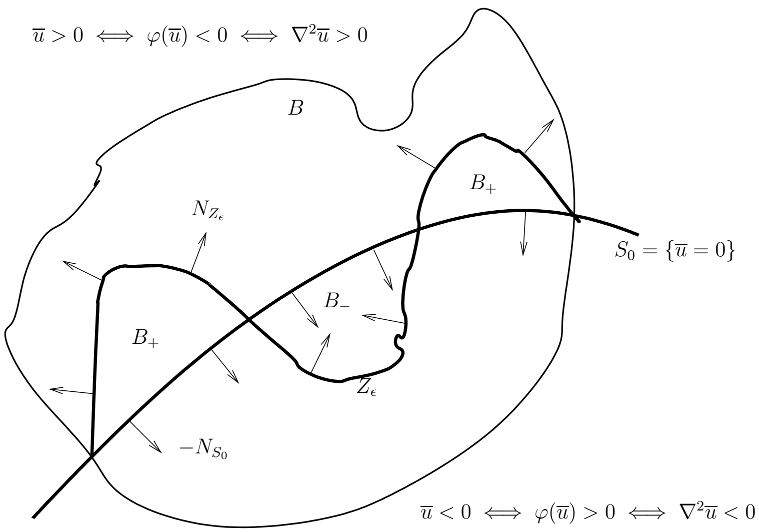

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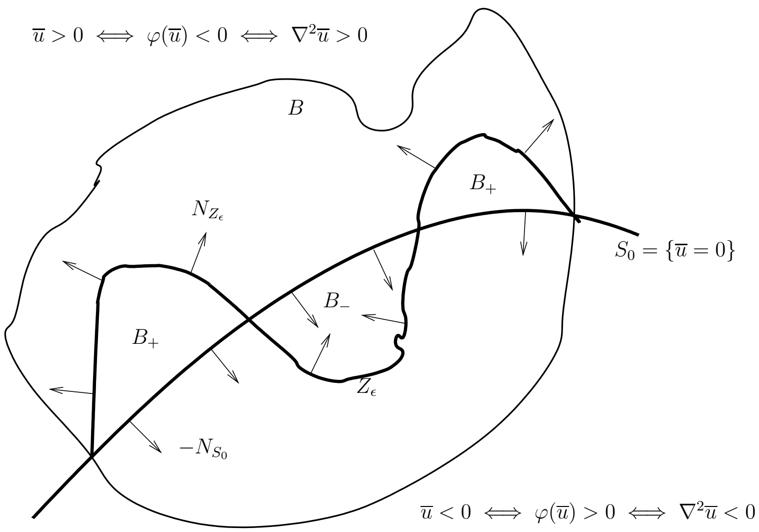

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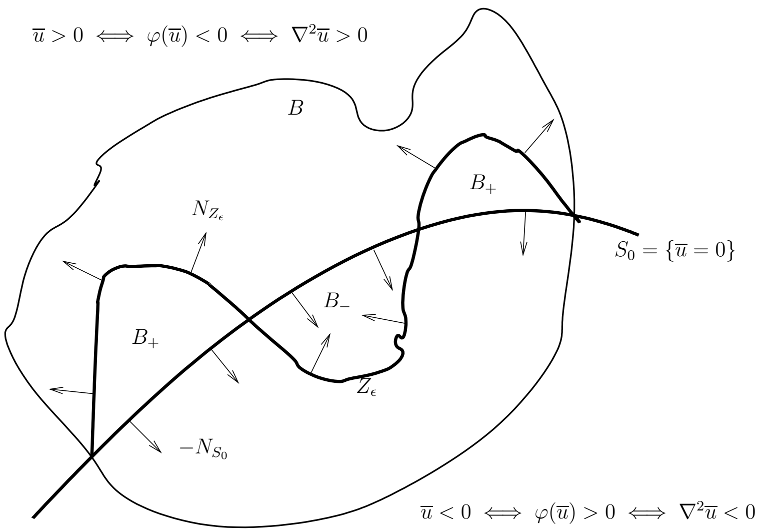

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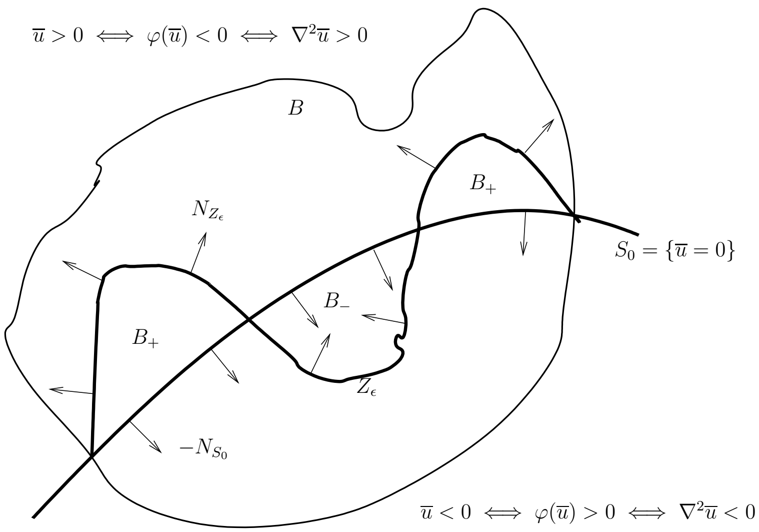

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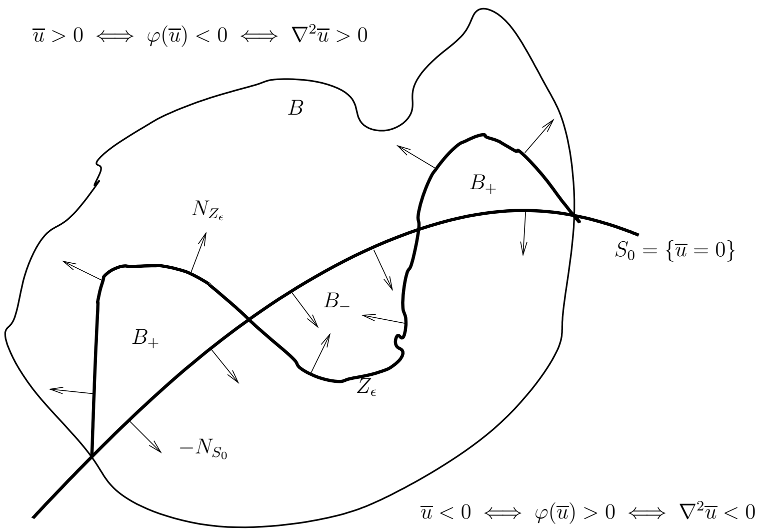

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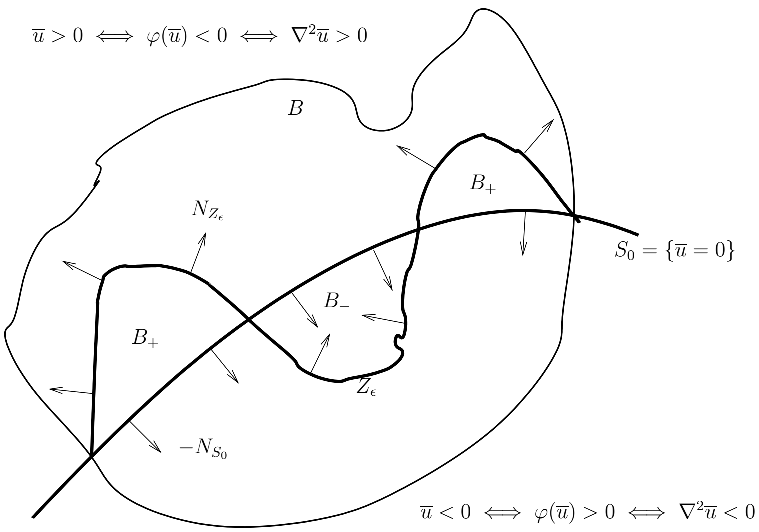

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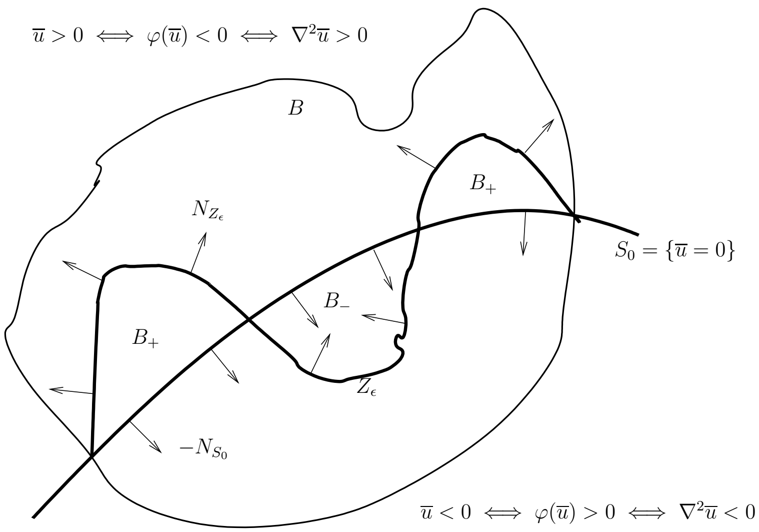

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(Recall that minimal surfaces are critical points of the area functional $S \mapsto \int_S dS$!)

Variational Principles for EC Surfaces V

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Then $\overline{u}$ is a critical point of $\mathcal{C}$. That is, i.e. $\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} \mathcal{C}(w_\varepsilon) = 0$.

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- Vary the chemical field. Then the Turing pattern has extremal diffusive flux through the EC surface.