Chaotic PDE's

Zbigniew Peradzyński

Institute of Applied Mathematics and Mechanics, Warsaw University, Banacha 2, 02-097 Warsaw, Poland
Institute of Fundamental Technological Research, Polish Academy of Science, Świętokrzyska 21, 00-049 Warsaw, Poland

It is generally believed that, chaos in dynamical systems is a consequence of nonlinearity. In the case of finite dimensional systems defined by systems of ordinary differential equations this indeed is true. However, as it is well known, the dynamics defined on $S^1 = \{R^1, \text{mod } 1\}$ by simple linear rule:

$$Y_{n+1} = 2\, Y_n \quad , \qquad \text{mod } 1$$

is chaotic, due to the nontrivial topology of S^1.

It turns out, that nonlinearity is not needed also in the case of dynamical systems defined by partial differential equations, hyperbolic as well as parabolic. There are various definitions of chaos. We adopt here the definition of Devaney [1].

1.Chaos (We use the definition of Devaney.)

X – separable metric space, $S_t : X \to X$ continuous semiflow

Def. $S_t : x \to x$ is chaotic if

1) Periodic points (orbits) are dense in x
2) S is transitive i.e. Any two neighbourhoods can be connected by an orbit
3) The flow has sensitive dependence on initial conditions

. **One can prove that under these conditions the dynamical system defined by (2) is chaotic i.e. every orbit, $t \to u(t,x)$, is unstable and there are orbits, which are dense in the whole phase space**

1. **Partial differential equations exhibiting chaos**

The simplest example of such an equation is given by:

$$u_t + au_x = \lambda u, \qquad \lambda > 0, \quad a > 0, \quad x \in (-\infty, 0),$$

another, more interesting for applications was proposed in 1977 by A. Lasota :

$$u_t + axu_x = \lambda u, \qquad x \in [0,1], \quad a > 0 \qquad (5)$$

and has chaotic solutions in the space $C^k(0,1)$ for $\lambda > k$!

Eq. (5) can be consider also in L^p-spaces, $p \in [1, \infty]$, of functions which are integrable with the power p. In L^p the chaos appears even for negative amplification coefficient λ, provided that

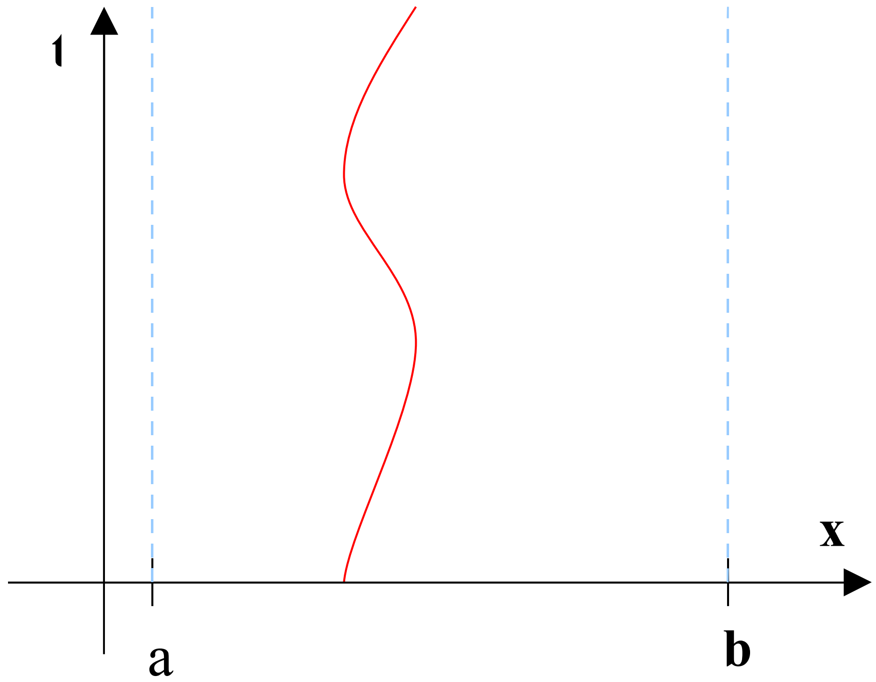
$$\lambda > -\frac{a}{p} \tag{6}$$

It turns out that under some conditions on the growth of f the results concerning Eq.(5) can be generalized to

$$u_t + c(t,x)u_x = f(t,x,u) \qquad x \in [a,b] \qquad (7)$$

provided that there exists a trapped characteristic.

The characteristic which stays all the time in [a,b] is called trapped characteristic.



We show also, that the quasilinear equation

$$u_t + c(u,x)u_x = f(t,x,u)$$

has chaotic solutions in some metric space X which is a certain open subset of space C^1. This shows that the results concerning chaos remain true when the semilinear equations are perturbed in such a way that the they become quasilinear. This is an important remark since generally we do the linearization procedure. To be sure that the properties of solutions of linearized equations are still valid in a fully nonlinear case we need such a result.

Finally, it can be shown that, these results can be generalized for hyperbolic system of equations, provided that:

a) there exists a trapped characteristics for at least one of waves existing in the system
b) this wave is amplified, when traveling along its characteristics, and the amplification coefficient at the trapped characteristic is large enough.

One may note, that in the problem of fluid (plasma) flow the trapped characteristic corresponds to the sonic line (i.e. passage from subsonic to supersonic flow).

PARABOLIC EQUATIONS

It is astonishing, that also the linear heat equation on the half line,

$$u_t + bu_x = u_{xx} + \lambda u, \qquad x \in [-\infty, 0], \quad \lambda > 0, \quad b > 0$$
$$u(0) = 0$$

with convective and a positive source terms, has chaotic solutions. Intuitively, this can be explained in a following way; The long waves of infinitely small amplitude (existing in the initial condition) are arriving from -∞ and are amplified. When reaching the vicinity of x=0, due to the boundary condition, they are converted into shorter waves, which are then dissipated. As a consequence the trajectory can be erratically wondering in the phase space.

Applications

Consider a linear strictly hyperbolic system

$$U_t + A(t,x)U_x = B(t,x)U \tag{1}$$

Let $\mu_1(t,x) < ... < \mu_n(t,x)$ be the eigenvalues of the matrix A. Let $X(t,x),...,X_n(t,x)$ and $L_1(t,x),...,L_n(t,x)$ be respectively the left and right eigenvectors corresponding to eigenvalues $\mu_1,...,\mu_n$. They can be normalized in such a way that $L_\alpha \circ X_\beta = \delta_{\alpha\beta}$.

Assuming the asymptotic form of the solution as

$$U = Y(t,x)e^{i\omega\psi(t,x)} + \sigma(\frac{1}{\omega}), \quad \omega \to \infty \tag{2}$$

one arrives at

$$i\omega(I\psi_t + \psi_x A)Y + Y_t + AY_x = BY$$

It follows that Y is an eigenvector, say $Y = \sigma X$ and $-\frac{\psi_t}{\psi_x}$ is the corresponding eigenvalue. Than the highest order part in ω is vanishing and from the rest we obtain an overdetermined system of equations for σ

$$(\sigma_t + \mu\sigma_x)X = \sigma(B - X_1 - AX_x) \tag{3}$$

Multiplying Eq. (3) by the left eigenvector we obtain finally

$$\sigma_t + \mu(t,x)\sigma_x = \sigma(B - X_t - \mu X_x) \tag{4}$$

which is a scalar equation for the wave amplitude σ – so called transport equation.

FLOW EQUATIONS

Example of a plasma flow in a Hall Thruster
-Assuming for simplicity that the electron temperature T_e is constant, the flow of plasma in the thruster is described by

$$
\begin{aligned}
n_t + (nu)_x &= \nu_i n \\
u_t + uu_x + c^2 (\ln n)_x &= \nu_e \left(\frac{I}{n} - u\right) - \nu_i u
\end{aligned}
\tag{8}
$$

where n, u – ion density and velocity, $c = \sqrt{\frac{kT_e}{m}}$ - ion sound velocity, m - ion mass, ν_i, ν_e - ionization and collision frequencies, $I(t)$ - total current density.

In Riemann invariants $R^+ = u + c \ln n, \quad R^- = u - c \ln n$ the system becomes

$$R_t^+ + (u+c)R_x^+ = v_e(\frac{I}{n} - u) + v_i(c-u)$$

$$R_t^- + (u-c)R_x^- = v_e(\frac{I}{n} - u) - v_i(c+u) \qquad (9)$$

which shows that indeed the linearized Eqs(9) as well as linearized equation for the R^-- wave have trapped characteristics.

This leads to the linear equation for small perturbation r of the invariant R^-

$$r_t + (u-c)r_x = \frac{1}{2}\left[v_e(\frac{I}{nc} - 1) - v_i - R^-_{0,x} \right] r \qquad (10)$$

$$u_t + v(x)u_x = f(u,x) \qquad , \quad x \in [0, L]$$

$$u(0,x) = u_0(x)$$

assuming that: *the given functions* $v(x)$, $f(u,x)$ *satisfy*:

1. $v(0) = 0,\ v(x) > 0,\ \text{for} \quad x > 0, \quad v'(0) > 0$,

2. $f(0,x) = 0,\ \text{ and } \ \dfrac{\partial f}{\partial u}(0,x) \geq \lambda > 0$

3. $\left| f(u,x) \right| \leq C + K\,|u|$

Cond.3 assures the existence of global in time solutions. One can prove that under these conditions the dynamical system defined by (2) is chaotic i.e. every orbit, $t \to u(t,x)$, is unstable and there are orbits, which are dense in the whole phase space $X = \left\{ u \in C(0,L);\ u(x=0) = 0,\ \| u \| = \sup_{x \in [0,L]} | u(x) \right\}$.

It is important to notice that $v(0,0)$ in (3a) can be replaced by $v(0,x^*) = 0$, for some $x^* \in (0,L)$. In other words we assume the existence of a trapped characteristic, i.e. the line $x(t)$ satisfying: $x'(t) = v(x),\ x(0) = x^*$, and such that it never approaches the right boundary of $[0,L]$.

References

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